JACOB'S LADDERS, ζ-FACTORIZATION AND INFINITE SET OF METAMORPHOSIS OF A MULTIFORM

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Abstract. In this paper we use Jacob's ladders together with fundamental Hardy-Littlewood formula (1921) to prove the so-called ζ-factorization formula on the critical line. Simultaneously, we obtain a set of control parameters of metamorphosis of a multiform connected with the Riemann-Siegel formula.

1. Introduction and main result

1.1. Let us remind the unique factorization theorem: any positive integer ($\neq 1$) can be expressed as a product of primes. This expression is unique except of the order in which the primes occur. Thus, we have

$$\begin{split} & n=p_1^{m_1}p_2^{m_2}\cdots p_r^{m_r},\ n\in\mathbb{N},\ n\neq 1, \\ & p_1<p_2<\dots<p_r,\ m_1,m_2,\dots,m_r\in\mathbb{N}, \end{split} \tag{1.1}$$

where

$$p_1,\dots,p_r$$

are primes. Next, Euler's identity (1737)

$$\sum_{n=1}^\infty \frac{1}{n^x}=\prod_p (1-p^{-x})^{-1},\ x>1 \tag{1.2}$$

may be regarded as an analytical equivalent of the unique factorization theorem.

Further, Riemann has defined in 1859 the zeta-function by the formula

$$\zeta(s)=\sum_{n=1}^\infty \frac{1}{n^s}=\prod_p (1-p^{-s})^{-1},\ s=\sigma+it\in\mathbb{C},\ \sigma>1, \tag{1.3}$$

and, consequently, by the method of analytic continuation, he defined the function $\zeta(s)$ at all finite points $s\in\mathbb{C}$ except for a simple pole at $s=1$. Now, since from (1.1)), (comp. (1.2), (1.3)) the exponential form of the factorization

$$\left(\frac{1}{n}\right)^x=\prod_{l=1}^r \left(\frac{1}{p_l^{m_l}}\right)^x,\ x>1, \tag{1.4}$$

follows (as an example), then we can formulate the following.

Remark 1. On the basis of the analytic continuation of the Riemann's zeta-function we may suppose that:

(a) there are inherited forms of the factorization (1.4) in the region $\sigma\leq 1,\ s\neq 1$, (together with mutation of these),

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(b) especially, we may suppose that on the critical line itself

$$\sigma = 1/2$$

there are some new species of the factorization property (1.4).

1.2. In this direction the following theorem holds true.

Theorem. Let

$$k = 1, \dots, k_0 \in \mathbb{N}$$

for every fixed k_0 and

$$H = H(T) \in \left(\frac{\ln\ln T}{\ln T}, T^{\frac{1}{\ln\ln T}}\right) \tag{1.5}$$

for every sufficiently big $T > 0$. Then for every k and for every H that fulfils condition (1.5) there are the functions

$$\begin{aligned} & H_k = H_k(T,H) > 0, \\ & \alpha_r = \alpha_r(T,H,k) > 0,\ r = 0,1,\dots,k, \\ & \alpha_r \neq \gamma:\ \zeta\left(\frac{1}{2} + i\gamma\right) = 0, \end{aligned} \tag{1.6}$$

such that the following ζ-factorization formula

$$\begin{aligned} & \sqrt{\frac{\Lambda}{\left|\zeta\left(\frac{1}{2} + i\alpha_0\right)\right|}} \sim \prod_{r=1}^{k} \left|\zeta\left(\frac{1}{2} + i\alpha_r\right)\right|, \\ & \Lambda = \Lambda(T,H,k) = \sqrt{2\pi}\frac{\sqrt{H}}{H_k}\ln^k T \end{aligned} \tag{1.7}$$

holds true. Moreover, the sequence

$$\{\alpha_r\}_{r=0}^{k}$$

has the following properties:

$$T < \alpha_0 < \alpha_1 < \dots < \alpha_k, \tag{1.8}$$

$$\alpha_{r+1} - \alpha_r \sim (1-c)\pi(T),\ r = 0,1,\dots,k-1, \tag{1.9}$$

where

$$\pi(T) \sim \frac{T}{\ln T},\ T \to \infty$$

is the prime counting function and c is the Euler's constant.

Remark 2. The asymptotic behavior of the set

$$\{\alpha_0, \alpha_1, \dots \alpha_k\} \tag{1.10}$$

by (1.9) is as follows: if $T \to \infty$ then the points of the set (1.10) recede unboundedly each from other and all these points together recede to infinity. Hence, at $T \to \infty$ the set (1.10) behaves like one dimensional Friedmann-Hubble expanding universe.

Remark 3. Consequently, the following holds true: even if the distances

$$\begin{aligned} & |\alpha_q - \alpha_r| \sim |q-r|(1-c)\pi(T) \to \infty \text{ as } T \to \infty, \\ & q, r = 0,1,\dots,k,\ q \neq r \end{aligned}$$

(see (1.9)) of elements of the set (1.10) are gigantic, there is a close constraint between the values of the set

$$\left\{\left|\zeta\left(\frac{1}{2}+i\alpha_0\right)\right|,\left|\zeta\left(\frac{1}{2}+i\alpha_1\right)\right|,\dots,\left|\zeta\left(\frac{1}{2}+i\alpha_k\right)\right|\right\},$$

namely the factorization formula (1.7).

2. On metamorphosis of a multiform generated by the Riemann-Siegel formula

2.1. Let us remind the Riemann-Siegel formula

$$\begin{aligned} &Z(t)=2\sum_{n\leq\tau(t)}\frac{1}{\sqrt{n}}\cos\{\vartheta(t)-t\ln n\}+\mathcal{O}(t^{-1/4}),\\ &\tau(t)=\sqrt{\frac{t}{2\pi}}, \end{aligned} \tag{2.1}$$

(see [6], p. 60, comp. [7], p. 79) where

$$\vartheta(t)=-\frac{t}{2}\ln\pi+\operatorname{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{1}{2}\right),$$

(see ([7]), p. 239). Next, we put in (2.1)

$$Z(t)=\sum_{n\leq\tau(t)}a_nf_n(t)+R(t), \tag{2.2}$$

where

$$\begin{aligned} &a_n=\frac{1}{\sqrt{n}},\\ &f_n(t)=\cos\{\vartheta(t)-t\ln n\},\\ &R(t)=\mathcal{O}(t^{-1/4}), \end{aligned} \tag{2.3}$$

and the functions $f_n(t)$ are nonlinear. That is, the function $Z(t)$ is nonlinear monoform. Now, we define the following multiform

$$G(x_1,\dots,x_k)=\prod_{r=1}^{k}|Z(x_r)|,\quad x_r>T>0,\ k\geq 2, \tag{2.4}$$

where the monoform $Z(t)$ is the generating function of G.

Further, we define the subset of all random sample points

$$\begin{aligned} &(\bar{x}_1,\dots,\bar{x}_k):\ T<\bar{x}_1<\bar{x}_2<\dots<\bar{x}_k,\\ &\bar{x}_r\neq\gamma:\ \zeta\left(\frac{1}{2}+i\gamma\right)=0,\ r=1,\dots,k, \end{aligned}$$

and propose the following

Question. Is there in this subset a point

$$(\bar{y}_1,\dots,\bar{y}_k)$$

of metamorphosis of the multiform (2.4) (that is, the point of significant change of the structure of the multiform (2.4))?

The answer is given in the following.

Corollary.

$$
\begin{aligned}
&\prod_{r=1}^{k}\left|\sum_{n\le\tau(\alpha_r)} a_n f_n(\alpha_r)+R(\alpha_r)\right| \sim \\
&\sim \sqrt{\frac{\Lambda}{\left|\sum_{n\le\tau(\alpha_0)} a_n f_n(\alpha_0)+R(\alpha_0)\right|}},\ T\to\infty,
\end{aligned} \tag{2.5}
$$

i.e. to the infinite subset of the points

$$\{\alpha_1(T),\alpha_2(T),\dots,\alpha_k(T)\},\quad T\in(T_0,\infty),$$

where T_0 is a sufficiently big, an infinite set of metamorphoses of the multiform (2.4) into quite distinct form on the right-hand side of (2.5) corresponds.

Remark 4. We shall call the elements of the set

$$\{\alpha_0(T),\alpha_1(T),\dots,\alpha_k(T)\},\ T\in(T_0,+\infty) \tag{2.6}$$

as *control parameters* (*functions*) of metamorphosis. The reason to this is that the parameters

$$\alpha_1(T),\dots,\alpha_k(T)$$

change the old form into the new one (see (2.5), and this last is controlled by the parameter $\alpha_0(T)$. That is, the set (2.6) plays a similar role as shem-ha-m'forash in Golem's metamorphosis.

3. On Hardy-Littlewood fundamental Lemma 18 from the memoir [2]

3.1. Let us remind the following (see [2], pp. 304, 305): if

$$
\begin{aligned}
J&=J(T,H)=\int_T^{T+U}\mathcal{J}^2\mathrm{d}t,\\
\mathcal{J}&=\mathcal{J}(t,H)=\int_t^{t+H}x(u)\mathrm{d}u,
\end{aligned} \tag{3.1}
$$

then the following Hardy-Littlewood formula

$$J=\int_T^{T+H}\mathcal{J}^2\mathrm{d}t=\pi\sqrt{2\pi}HU+\mathcal{O}\left(\frac{U}{\ln T}\right) \tag{3.2}$$

holds true, where

$$U\in[T^a,T^b],\ \frac12<a<b\le\frac58,\ 0<H\le T^\epsilon, \tag{3.3}$$

and ϵ is positive and sufficiently small, and (see [2], p. 290)

$$\zeta\left(\frac12+it\right)=-\left(\frac{2}{\pi}\right)^{1/4}e^{i\pi/8}(2\pi e)^{\frac12 it}e^{-\frac12 it\ln t}x(t)\left\{1+\mathcal{O}\left(\frac1t\right)\right\}. \tag{3.4}$$

3.2. We use the Riemann function $Z(t)$ instead of the Hardy-Littlewood function $x(t)$. Namely, the formula

$$\zeta\left(\frac{1}{2}+it\right)=e^{-i\vartheta(t)}Z(t) \tag{3.5}$$

together with formula for $\vartheta(t)$ from [7], p. 329 give that

$$x(t)=-\left(\frac{\pi}{2}\right)^{1/4}Z(t)\left\{1+\mathcal{O}\left(\frac{1}{t}\right)\right\}. \tag{3.6}$$

Hence, we use the following variant of the Hardy-Littlewood formula (see (3.6))

$$\bar{J}=\bar{J}(T,H)=\int_T^{T+U_0}\bar{\mathcal{J}}^2\mathrm{d}t=2\pi HU_0+\mathcal{O}\left(\frac{U_0}{\ln T}\right), \tag{3.7}$$

where (comp. (3.1) – (3.3))

$$\begin{aligned} & \bar{\mathcal{J}}=\bar{\mathcal{J}}(t,H)=\int_t^{t+H}Z(u)\mathrm{d}u, \\ & U_0=T^{0.5001}, \\ & H\in\left(\frac{\ln\ln T}{\ln T},T^{\frac{1}{\ln\ln T}}\right). \end{aligned} \tag{3.8}$$

Remark 5. Since (see (3.7))

$$2\pi HU_0+\mathcal{O}\left(\frac{U_0}{\ln T}\right)=2\pi HU_0\left\{1+\mathcal{O}\left(\frac{1}{H\ln T}\right)\right\},$$

then we have that the formula (3.7) is asymptotic formula for H of (3.8).

3.3. Let us remind the following sentences of Hardy and Littlewood (see [2], p. 315): *As was observed in 5.5, we do not use the full force of Lemma 18. The complete lemma, however, seems of considerable interest in itself, and it may prove to be of service in the future.*

Remark 6. We notice explicitly, that our theory of the Jacob's ladders together with the Hardy-Littlewood asymptotic formula (3.7) constitute the basis of our result about metamorphosis of corresponding multiform. Thus, we have obtained, after 94 years , the result that use the full force of the asymptotic Hardy-Littlewood formula from their Lemma 18.

4. JACOB'S LADDERS AND THE HARDY-LITTLEWOOD INTEGRAL (1918)

Let us remind that we have introduced (see [4], (9.1), (9.2)) the following formula

$$\tilde{Z}^2(t)=\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t}, \tag{4.1}$$

where

$$\begin{aligned} & \tilde{Z}^2(t)=\frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]}=\frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^2}{\omega(t)}, \\ & \omega(t)=\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t. \end{aligned} \tag{4.2}$$

The function

$$\varphi_1(t)$$

is called the Jacob's ladder (see our paper [3]) according to Jacob's dream in Chumash, Bereishis, 28:12, has the following properties:

(a)
$$\varphi_1(t)=\frac{1}{2}\varphi(t),$$

(b) the function $\varphi(t)$ is a solution of the nonlinear integral equation (see [3], [4])
$$\int_0^{\mu[x(T)]} Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t=\int_0^T Z^2(t)\mathrm{d}t,$$
where each admissible function $\mu(y)$ generates the solution
$$y=\varphi_\mu(T)=\varphi(T),\ \mu(T)\geq 7y\ln y.$$

Remark 7. The main goal of introducing the Jacob's ladders is described in [3], where we have shown, by making use of these Jacob's ladders, that the Hardy-Littlewood integral (1918)
$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t$$
(see [1], pp. 122, 151 – 156) has - in addition to the Hardy-Littlewood expression (and also other similar to this one) possessing an unbounded error at $T\to\infty$ - the following infinite set of almost exact expressions
$$\begin{aligned}&\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2\mathrm{d}t=\varphi_1(T)\ln\varphi_1(T)+(c-\ln 2\pi)\varphi_1(T)+\\&+c_0+\mathcal{O}\left(\frac{\ln T}{T}\right),\quad T\to\infty,\end{aligned}$$
where c_0 is the constant from the Titchmarsh-Kober-Atkinson formula (see [7], p. 114).

Remark 8. The Jacob's ladder $\varphi_1(T)$ can be interpreted by our formula (see [3], (6.2))
$$T-\varphi_1(T)\sim(1-c)\pi(T),$$
where $\pi(T)$ is the prime-counting function, as an asymptotic complementary function to the function
$$(1-c)\pi(T)$$
in the following sense
$$\varphi_1(T)+(1-c)\pi(T)\sim T,\quad T\to\infty.$$

5. Proof of Theorem

5.1. First of all we obtain from the formula
$$\int_T^{T+U_0}\left\{\int_t^{t+H}Z(u)\mathrm{d}u\right\}^2\mathrm{d}t\sim 2\pi HU_0$$
(see (3.7), (3.8)), by mean-value theorem, that
$$\left\{\int_\eta^{\eta+H}Z(u)\mathrm{d}u\right\}^2\sim 2\pi H,$$

i.e.

$$\left|\int_{\eta}^{\eta+H} Z(u)\mathrm{d}u\right| \sim \sqrt{2\pi H},\ \eta=\eta(T,H)\in(T,T+U_0). \tag{5.1}$$

Since (see (3.8))

$$H=o\left(\frac{T}{\ln T}\right), \tag{5.2}$$

then by our lemma (comp. [5], (7.1), (7.2))

$$\left|\int_{\overset{k}{\eta}}^{\overset{k}{\widehat{\eta+H}}} Z[\varphi_1^k(t)]\prod_{r=0}^{k-1}\tilde{Z}^2[\varphi_1^r(t)]\mathrm{d}t\right| \sim \sqrt{2\pi H}$$

and, by the mean-value theorem, we obtain that

$$\begin{gathered} |Z[\varphi_1^k(\beta)]| \sim \sqrt{2\pi}\frac{\sqrt{H}}{\overset{k}{\widehat{\eta+H}}-\overset{k}{\eta}}\prod_{r=0}^{k-1}\tilde{Z}^{-2}[\varphi_1^r(\beta)], \\ \beta=\beta(T,H,k)\in(\overset{k}{\eta},\overset{k}{\widehat{\eta+H}}). \end{gathered} \tag{5.3}$$

5.2. Now, we make a transformation of the formula (5.3). First of all, we have (comp. [5], Property 2, (6.4))

$$\beta\in(\overset{k}{\eta},\overset{k}{\widehat{\eta+H}}) \ \Rightarrow\ \varphi_1^r(\beta)\in(\overset{k-r}{\eta},\overset{k-r}{\widehat{\eta+H}}),\ r=0,1,\dots,k,$$

i.e.

$$\begin{aligned} &\varphi_1^0(\beta)=\alpha_k\in(\overset{k}{\eta},\overset{k}{\widehat{\eta+H}}), \\ &\varphi_1^1(\beta)=\alpha_{k-1}\in(\overset{k-1}{\eta},\overset{k-1}{\widehat{\eta+H}}), \\ &\vdots \\ &\varphi_1^{k-2}(\beta)=\alpha_2\in(\overset{2}{\eta},\overset{2}{\widehat{\eta+H}}), \\ &\varphi_1^{k-1}(\beta)=\alpha_1\in(\overset{1}{\eta},\overset{1}{\widehat{\eta+H}}), \\ &\varphi_1^{k}(\beta)=\alpha_0\in(\eta,\eta+H). \end{aligned} \tag{5.4}$$

Consequently, from (5.3) by (5.4), (4.2) the formula

$$\begin{gathered} \left|\zeta\left(\frac{1}{2}+i\alpha_0\right)\right| \sim \sqrt{2\pi}\frac{\sqrt{H}}{H_k}\prod_{l=1}^{k}\frac{\omega(\alpha_l)}{\left|\zeta\left(\frac{1}{2}+i\alpha_l\right)\right|^2}, \\ \alpha_r=\alpha_r(T,H,k),\ r=0,1,\dots,k,\ H_k(T,H)=\overset{k}{\widehat{\eta+H}}-\overset{k}{\eta} \end{gathered} \tag{5.5}$$

follows.

5.3. Next, let us remind the following properties of the disconnected set

$$\Delta(T,H,k)=\bigcup_{r=0}^{k}[\overset{r}{\eta},\overset{r}{\widehat{\eta+H}}].$$

(see (5.2), comp. [5], (2.5) – (2.7), (2.9)) . If

$$H=o\left(\frac{T}{\ln T}\right),$$

then

$$\begin{aligned}
& |[\overset{r}{\eta},\overset{r}{\widehat{\eta+H}}]|=\overset{r}{\widehat{\eta+H}}-\overset{r}{\eta}=o\left(\frac{T}{\ln T}\right),\ r=1,\dots,k, \\
& |[\overset{r-1}{\widehat{\eta+H}},\overset{r}{\eta}]|\sim(1-c)\pi(T), \qquad (5.6) \\
& [\eta,\eta+H]\prec[\overset{1}{\eta},\overset{1}{\widehat{\eta+H}}]\prec\dots\prec[\overset{k}{\eta},\overset{k}{\widehat{\eta+H}}].
\end{aligned}$$

Hence, from (5.4), (5.6) the properties (1.8), (1.9) follow immediately. Further, we have (comp. [5], (4.3))

$$\ln t\sim\ln\eta,\ \forall t\in(\eta,\overset{k}{\widehat{\eta+H}}),$$

and (see (3.8), (5.1))

$$\ln t\sim\ln\eta\sim\ln T,\ \forall t\in(\eta,\overset{k}{\widehat{\eta+H}}). \tag{5.7}$$

Consequently, we obtain from (5.5) by (4.2), (5.7) the following

$$\left|\zeta\left(\frac12+i\alpha_0\right)\right|\sim\sqrt{2\pi}\frac{\sqrt{H}}{H_k}\ln^k T\prod_{r=1}^{k}\left|\zeta\left(\frac12+i\alpha_r\right)\right|^{-2},$$

i.e. the formula (1.7) holds true.

6. Concluding remarks

If we use the formulae (comp. [7], pp. 221, 329)

$$\begin{aligned}
& \vartheta(t)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\mathcal{O}\left(\frac1t\right), \\
& \vartheta'(t)=\frac12\ln\frac{t}{2\pi}+\mathcal{O}\left(\frac1t\right), \\
& \vartheta''(t)\sim\frac{1}{2t}, \\
& t\to\infty
\end{aligned}$$

then we obtain from (2.1), by a little of transformations, the local kind of the spectral representation

$$\begin{aligned}
& Z(x_r)=2\sum_{n\le\tau(x_r)}\frac{1}{\sqrt{n}}\cos\left\{t\ln\frac{\tau(x_r)}{n}-\frac{x_r}{2}-\frac{\pi}{8}\right\}+ \\
& +\mathcal{O}(x_r^{-1/4}), \qquad (6.1) \\
& t\in[x_r,x_r+H],H\in\left(\frac{\ln\ln T}{\ln T},T^{\frac{A}{\ln\ln T}}\right),\ \tau(x_r)=\sqrt{\frac{x_r}{2\pi}},
\end{aligned}$$

i.e. the Riemann-Siegel formula (2.1), $t=x_r$.

Remark 9. Namely, the sequence

$$\{\omega_{n,r}\}_{n\leq\tau(x_r)},\ \omega_{n,r}=\ln\frac{\tau(x_r)}{n},\ r=1,\dots,k \tag{6.2}$$

of the cyclic frequencies $\omega_{n,r}$ will be called as the local spectrum of the Riemann-Siegel formula (2.1).

Remark 10. Consequently, we have (see (2.4), (6.1), (6.2)) that the multiform

$$G(x_1,x_2,\dots,x_k)$$

expresses the complicated oscillating process.Just for this oscillating multiform we have constructed the set of metamorphosis described by the formula (2.5).

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